

Strong-coupling behaviour in discrete Kardar - Parisi - Zhang equations

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1996 J. Phys. A: Math. Gen. 29 7917

(<http://iopscience.iop.org/0305-4470/29/24/016>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.71

The article was downloaded on 02/06/2010 at 04:07

Please note that [terms and conditions apply](#).

Strong-coupling behaviour in discrete Kardar–Parisi–Zhang equations

T J Newman and A J Bray

Department of Physics, University of Manchester, Manchester M13 9PL, UK

Received 15 April 1996, in final form 15 October 1996

Abstract. We present a systematic discretization scheme for the Kardar–Parisi–Zhang (KPZ) equation, which correctly captures the strong-coupling properties of the continuum model. In particular we show that the scheme contains no finite-time singularities in contrast to conventional schemes. The implications of these results to (i) previous numerical integration of the KPZ equation, and (ii) the non-trivial diversity of universality classes for discrete models of ‘KPZ-type’ are examined. The new scheme makes the strong-coupling physics of the KPZ equation more transparent than the original continuum version and allows the possibility of building new continuum models which may be easier to analyse in the strong-coupling regime.

1. Introduction

Within the field of non-equilibrium interface growth, much attention has been given to a continuum Langevin description first suggested by Kardar, Parisi and Zhang (KPZ) [1, 2]. The KPZ equation for the interface profile $h(\mathbf{x}, t)$ takes the form

$$\partial_t h = \nu \nabla^2 h + \lambda (\nabla h)^2 + \eta \quad (1)$$

where $\eta(\mathbf{x}, t)$ is a Gaussian white noise with zero mean. One is interested in asymptotic scaling behaviour of this model. In particular the steady-state two-point correlation function $C(|\mathbf{r} - \mathbf{r}'|, t - t') \equiv \langle (h(\mathbf{r}, t) - h(\mathbf{r}', t'))^2 \rangle$ is expected to assume a scaling form

$$C(R, \tau) \sim R^{2\chi} f(R^z/\tau) \quad (2)$$

where χ and z are the roughness and dynamic exponents respectively. There exists a scaling relation between these exponents of the form $\chi + z = 2$. The determination of these exponents along with the scaling function $f(y)$ are the prime goals in this field. For substrate dimension $d \leq 2$ the exponents take on non-trivial values for all $\lambda \neq 0$. For $d > 2$ there exists a phase transition between smooth and rough phases. For small λ the interface is smooth ($\chi = 0$)—this behaviour is described by the so-called weak-coupling fixed-point. For large λ the interface is asymptotically rough and χ takes on a non-trivial d -dependent value—this is the physics of the strong-coupling (SC) fixed point.

It is convenient for us to mention at this point the interesting connection between the KPZ equation and the physics of directed polymers in a random medium. This may be realized by making the Hopf–Cole transformation $h(\mathbf{x}, t) = (\nu/\lambda) \ln w(\mathbf{x}, t)$ which yields the equation

$$\partial_t w = \nu \nabla^2 w + (\lambda/\nu) w \eta. \quad (3)$$

The field w is interpreted as the restricted partition function of the directed polymer (which is more clear if one rewrites the above equation as a Feynman path integral). Although the equation is linear, the noise is now multiplicative which forbids any simple analytic progress.

In fact there are no systematic methods available to study the SC behaviour of the KPZ equation analytically. The most useful technique to date is a mode-coupling scheme [3] which certainly gives non-trivial behaviour in higher dimensions, but suffers from the draw-back that it is very difficult to correct systematically. (In particular the mode-coupling scheme indicates $d = 4$ as an upper critical dimension above which the exponents take on mean-field values; i.e. $\chi = 0$.)

In such a situation, numerical methods take on a supreme significance. In attempting to understand the KPZ model, two main numerical methods have been employed. The first is that of simulating microscopically motivated models which are believed on physical grounds to lie within the KPZ universality class. Such models include Eden growth [4], polynuclear growth models [5], the hypercube stacking model [6] and the Kim–Kosterlitz model [7]. The dynamical exponents measured for these different models all lie within some small range, although numerical precision is not good enough (especially in higher dimensions) to give a definitive answer as to whether these models share the same universal features. The more subtle question of whether their universal features are actually those of the KPZ equation, relies upon measuring the exponents (and in principle the scaling functions) for the KPZ equation itself. The lack of analytic techniques means that such measurements may only be made by direct numerical integration of a discretized form of the KPZ equation. (Alternatively one may perform numerical studies of the directed polymer analogue via transfer matrix methods.) Such a numerical procedure has been carried out by several groups over the years [8–10] with varying degrees of success. A common feature of these attempts is that numerical integration becomes increasingly difficult as the value of the coupling constant is raised. (The instability for large coupling was also noted in [11].) This is unfortunate as ideally one would like to be deep inside the SC region, which for higher dimensions implies that the dimensionless coupling constant is very large. In this paper we shall show that the choice of spatial discretization of the KPZ equation is crucial in order to ensure stability under integration, and more importantly to ensure that the discrete model is still within the universality class of the original continuum model. The discretization schemes used in previous studies unfortunately failed on both these points and the results so obtained must be viewed with scepticism.

The remainder of this paper takes the following form. In the next section we study, in some detail, the conventional discretization schemes used to mimic the continuum KPZ equation. We shall show that they contain defects such as instabilities and ghost fixed points which are absent in the continuum. In section 3 we motivate a new discretization scheme which is guaranteed to be stable and which also contains no unphysical fixed points. This model is the ideal basis for a fresh numerical investigation of the KPZ equation. In section 4 we look at the SC properties of this discrete model and find that it is physically more transparent than its continuum counterpart. Using this new insight we make some speculative propositions concerning new continuum models which are intrinsically SC versions of KPZ. The hope is that these models are more analytically susceptible than the KPZ equation. We end with section 5, in which we discuss the nature of universality in the space of discrete ‘KPZ-type’ models, along with giving our conclusions.

2. Conventional discretization schemes

In order to perform numerical integration of the KPZ equation it is necessary to discretize both space and time. In this paper we shall be exclusively concerned with the delicate nature of the spatial discretization. Temporal discretization is a less challenging problem, and there exist many techniques (such as predictor-corrector methods) which help to stabilize the integration. So, henceforth, we will keep time continuous. Now the standard spatial discretization for the KPZ equation is performed as follows. One replaces the function $h(\mathbf{x}, t)$ by a set of fields $h_i(t)$ where the subscript is a lattice index. The lattice is taken to be hypercubic with spacing a . One then rewrites equation (1) as

$$dh_i(t)/dt = (\nu/a^2) \sum_j (h_j(t) - h_i(t)) + (\lambda/2a^2) \sum_j (h_j(t) - h_i(t))^2 + \eta_i(t) \quad (4)$$

where the sums are over nearest-neighbour sites.

There is a minor subtlety here which we shall address immediately before continuing. In writing the discrete form of $(\nabla h)^2$ there are two obvious choices. The one favoured in past work [8–10] takes the form (where $d = 1$ for simplicity)

$$(\partial h/\partial x)^2 \rightarrow (1/4a^2)(h_{i+1} - h_{i-1})^2 \quad (5)$$

whilst the alternative is

$$(\partial h/\partial x)^2 \rightarrow (1/2a^2)[(h_{i+1} - h_i)^2 + (h_i - h_{i-1})^2]. \quad (6)$$

At this level of the discussion we state that the second form is preferable, as the first form has the undesirable *lattice* feature that it vanishes when the height at site i is a symmetric peak. Studying surface morphologies generated by the first form shows that such sites get ‘left behind’ by their neighbours, as this site receives no driving force from the nonlinearity. This point is academic in the light of the remaining discussion in this section.

We shall take two approaches in order to illustrate the problems with the above discretization scheme. Both approaches are concerned with the deterministic version of the problem, and both yield the result that even in the absence of noise the above discretization scheme is unstable, which is entirely unphysical as the deterministic continuum model is completely stable and has a single large time solution of $h = \text{constant}$ (for bounded initial data). In the first approach we compare the numerically integrated solution of equation (4) with the exact solution of the deterministic continuum equation, for the simple initial condition of a top hat. In the second approach we exactly solve the discrete equation for some simple cases and show explicitly that the solutions are unstable. The inherent instability becomes progressively worse as the coupling λ is increased in value.

So, beginning with the first approach we consider the deterministic version of the KPZ equation. An exact solution of this equation may be obtained via the Hopf–Cole transformation and takes the form

$$h(\mathbf{x}, t) = (\nu/\lambda) \ln \left\{ \int d^d y g(\mathbf{x} - \mathbf{y}, t) \exp[\lambda h(\mathbf{y}, 0)/\nu] \right\} \quad (7)$$

where $g(\mathbf{x}, t)$ is the heat kernel.

We consider $d = 1$ for simplicity. Consider the simple initial condition of a top hat of height H and width $2b$ centred at the origin. The above solution then simplifies to

$$h(x, t) = \frac{\nu}{\lambda} \ln \left\{ 1 + \frac{(\exp(\lambda H/\nu) - 1)}{2} \left[\operatorname{erf} \left(\frac{b-x}{(4\nu t)^{1/2}} \right) + \operatorname{erf} \left(\frac{b+x}{(4\nu t)^{1/2}} \right) \right] \right\} \quad (8)$$

where $\operatorname{erf}(z)$ is the error function [12]. It is easy to convince oneself that $h(x, t)$ is a monotonically decreasing function of time for $|x| \leq b$. In other words, the block decays

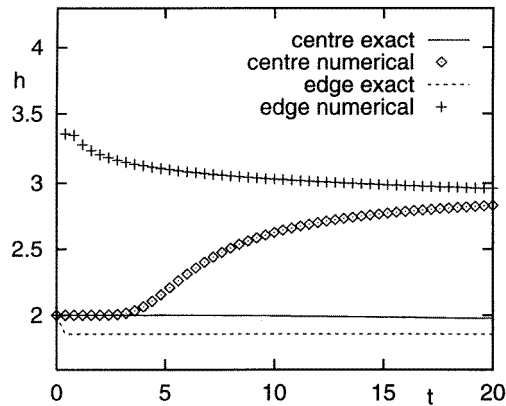


Figure 1. The time evolution of the centre of the top hat function, and of one edge of the top hat function, for $\lambda = 10\lambda_c$. The curves represent the exact solution as given in equation (8), and the points represent numerical integration using the conventional discretization scheme shown in equation (4). Parameter values are $\nu = 1.0$, $a = 1.0$, $b = 10.0$, $H = 2.0$ and for the numerical integration $\Delta t = 0.002$. For these values, $\lambda_c = 0.5$.

away to zero, for any size of the coupling constant λ . If one now uses equation (4) to evolve the top hat function one finds that this smooth decay only occurs when λ is small. In fact the value of the coupling where unphysical behaviour sets in is of order $\lambda_c = \nu/H$. In figure 1 we plot the evolution of the height at the centre of the top hat and at one edge as a function of time, for $\lambda = 10\lambda_c$ —curves corresponding to the exact solution given in equation (8), and direct numerical integration of the discrete scheme of equation (4) are shown. One sees that the discrete equation fails completely to capture the correct evolution. This effect cannot be corrected by reducing the time step. It is a consequence of the spatial discretization and would occur even for continuous time.

This result is very disappointing as λ_c demarcates the transition between weak and SC behaviour for this simple problem. From the exact solution one sees that interesting and revealing KPZ physics is present. For $\lambda \ll \lambda_c$ the nonlinearity is essentially unimportant and the block diffuses away. More precisely, the centre of the block decays as

$$h(0, t) \sim \frac{bH}{(\pi \nu t)^{1/2}}. \quad (9)$$

This power-law decay also holds for $\lambda \gg \lambda_c$ but only for *extremely* large times. When the coupling exceeds its critical value, a new time scale t^* emerges, which takes the form $t^* = (b^2/\pi \nu) \exp(2\lambda/\lambda_c)$. The block diffuses away only for $t \gg t^*$. For $t < t^*$ the centre of the block is essentially frozen, due to the nonlinearity pushing out the sides of the block. In fact the decay of the centre in this time regime is of the form

$$h(0, t) \sim H - (v/2\lambda) \ln(\pi \nu t/b^2). \quad (10)$$

In appendix A we discuss in more detail the relevance of this result to the SC physics of the noisy KPZ equation.

The discrete equation fails for the top hat initial condition due to the large height deviation which occurs over a few lattice sites. We have tested the discrete equation on smooth initial conditions such as an inverted parabola and a Gaussian, and it reproduces the correct evolution. In the stochastic version of the model, the noise is constantly producing discontinuities into the interface, so any discrete scheme must be stable in the presence of large, local height deviations.

So to summarize the result of the first approach, we see that even for a simple scenario—that of the deterministic evolution of a block profile—the discrete equation fails to describe the SC physics.

We now turn to the second approach—that of solving the discrete equation (4) explicitly (again in the deterministic version) for some simple situations. As a warm-up we consider the ostensibly artificial problem of just three sites on a ring. We are led to consider the coupled equations

$$\begin{aligned} dh_1/dt &= h_3 + h_2 - 2h_1 + (\lambda/2)[(h_1 - h_3)^2 + (h_2 - h_1)^2] \\ dh_2/dt &= h_1 + h_3 - 2h_2 + (\lambda/2)[(h_2 - h_1)^2 + (h_3 - h_2)^2] \\ dh_3/dt &= h_2 + h_1 - 2h_3 + (\lambda/2)[(h_3 - h_2)^2 + (h_1 - h_3)^2] \end{aligned} \quad (11)$$

where we have set $v = 1 = a$. Eliminating the soft mode (the mean height) we have two independent modes which we combine as $u_1 = (\lambda/6)(h_1 - h_2)$ and $u_2 = (\lambda/6)(h_2 - h_3)$. The above equations now read (after rescaling $t \rightarrow t/3$)

$$\begin{aligned} du_1/dt &= u_1(-1 + u_1 + 2u_2) \\ du_2/dt &= u_2(-1 - u_2 - 2u_1). \end{aligned} \quad (12)$$

The alternative discretization of $(\nabla h)^2$ yields the same equations as above, but with $\lambda \rightarrow -2\lambda$. Given this is a deterministic system, we expect on the basis of the deterministic KPZ equation for the final state to be that of all three heights equal, independent of the initial condition, i.e. $u_i(t) \rightarrow 0$. That this is not the case for this discrete model may be seen by finding the fixed points of equations (12), along with their associated stability. There are actually four fixed points, only one of which is the physically reasonable one. The remaining three fixed points correspond to one particular configuration, allowing for the threefold translational degeneracy. In figure 2 we show the dynamic flows of the above equations and sketch the approximate boundaries of stability. The insets show the configurations relating to the various fixed points. One sees that the model is only driven to the correct fixed point for initial conditions which lie within the boundaries of stability. The linear dimensions of the region (in terms of the original h_i variables) are of order $1/\lambda$. This indicates once again that as λ is increased the stability of the discrete equations decreases.

Although this problem seems far from any interesting situation, this is not the case. Since we have enforced periodic boundary conditions, the above result shows that any periodic chain whose length is a multiple of three will be unstable since the chain only has to partition itself into groups of three for the unstable fixed points found above to become active. Of course, as the number of particles increases, there will be an increasingly large number of fixed points corresponding to different modes of the system. That there exists at least one such unstable mode (as we have demonstrated above) is sufficient for our purposes. This result states that the discrete scheme given in equation (4) is intrinsically unstable due to the existence of ghost (i.e. unphysical) fixed points which separate the evolution of the system from its true asymptotic state.

To illustrate this point further, we briefly consider one more model solution of the discrete equations, this time for an arbitrarily large number of sites. Consider equation (4) for N height variables $\{h_i(t)\}$ coupled together via the scheme given in equation (5). (For variety, we use the alternative discretization of $(\nabla h)^2$.) We take the ends of the chain to be pinned; i.e. $h_0(t) = 0 = h_{N+1}(t)$. Again, the physics of the continuum problem admits only one asymptotic solution—that of a flat interface with $h_i(t) = 0$. The explicit form of

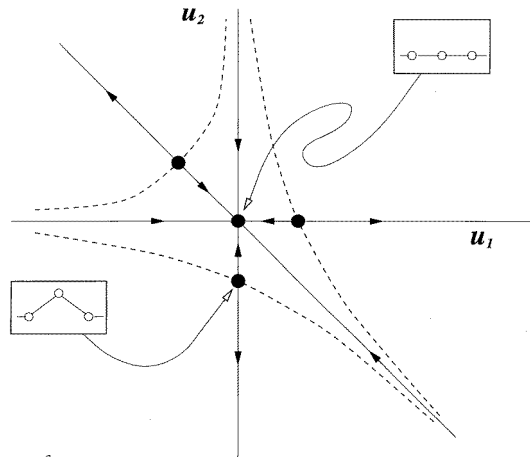


Figure 2. The phase space of the three-site problem. The broken curves represent the domain of stability which has linear dimensions of $O(1/\lambda)$. The insets indicate the site configurations corresponding to the various fixed points.

the discrete equations is

$$\begin{aligned} dh_1/dt &= -2h_1 + h_2 + (\lambda/4)h_2^2 \\ dh_i/dt &= h_{i-1} - 2h_i + h_{i+1} + (\lambda/4)(h_{i+1} - h_{i-1})^2 \quad 2 \leq i \leq N-1 \\ dh_N/dt &= h_{N-1} - 2h_N + (\lambda/4)h_{N-1}^2. \end{aligned} \quad (13)$$

The task of finding all the fixed points of the above equations is extremely difficult. We will content ourselves with studying an obvious fixed point: making the ansatz $h_i(t) = h(t)$, we see that such a solution picks out two fixed points. The first is the physical one $h(t) = 0$, whilst the second is $h(t) = 4/\lambda$. An analysis of the physical fixed point reveals, of course, its *linear* stability. The stability analysis of the unphysical fixed point follows from writing $h_i(t) = 4/\lambda + \hat{h}_i(t)$. Keeping linear terms only, we have the stability equations

$$d\hat{h}_1/dt = -2\hat{h}_1 + 3\hat{h}_2 \quad (14)$$

$$d\hat{h}_i/dt = \hat{h}_{i-1} - 2\hat{h}_i + \hat{h}_{i+1} \quad 2 \leq i \leq N-1 \quad (15)$$

$$d\hat{h}_N/dt = 3\hat{h}_{N-1} - 2\hat{h}_N. \quad (16)$$

The stability criterion is now reduced to the following question. Does the matrix \mathbf{M} defined by $d\hat{\mathbf{h}}/dt = \mathbf{M}\hat{\mathbf{h}}$ have any positive eigenvalues, and if so, how do these eigenvalues depend on N . Details of a numerical solution are given in appendix B. We find the following result: for $N \leq 6$ there exists one positive eigenvalue, whereas for $N > 6$ there exist two positive eigenvalues which become degenerate and of order unity, in the limit of large N . We therefore see that this N body problem is also unstable and once again, the instability occurs for height deviations which become greater than a value of $O(1/\lambda)$.

We shall leave a full discussion of these results for the final section. At this point we just re-iterate the results of this section. We have studied the deterministic version of the conventional discretization scheme from a number of viewpoints. It has been found to yield both unphysical fixed points, and associated instability, whenever neighbouring height differences exceed a critical value of $O(1/\lambda)$. *Thus, it has no connection to the continuum model in the SC regime.* The addition of noise to this scheme only decreases its range

of applicability since (even for weak noise) strong fluctuations will create critical height differences thus driving the evolution away from the correct asymptotic form.

3. New discretization scheme

The previous section has established the entirely negative result that conventional discretization schemes of the KPZ equation are in fact physically unrelated to that equation, with the degree of physical integrity decreasing as one increases the coupling constant. The important question remains—how does one discretize the KPZ equation in such a way as to ensure stability and also to retain the important physics? In this section we provide an answer to this question which turns out to be extremely simple, but also physically revealing.

Before doing so, however, we consider the possibility that the discrete scheme displayed in equation (4) can be regularized by the addition of further terms, thereby avoiding any instability. Such an approach has recently been developed in the context of an exactly soluble mean-field model [13]. In its simplest form, it corresponds to the addition of a term $\kappa(\nabla h)^2 \nabla^2 h$ to the right-hand side of equation (1), followed by a ‘naive discretization’ as in equation (4). The resulting equation is soluble in the infinite-range limit where each height h_i is coupled to every other height and, in the limit of weak noise, exhibits an interesting phase transition as a function of the variable $g = \lambda/\sqrt{\kappa v}$. The SC phase $g > g_c$ is characterized by a double-peaked distribution for the fluctuations of the local heights h_i from the mean height [13]. This implies bumps of a definite height in the surface, reminiscent of numerical studies of a similar regularized model [14]. This behaviour is, however, quite different from that of the supposedly related directed polymer model, leading one to question whether, even after regularization, the discrete and continuum models belong to the same universality class. This is the question we now address.

As we mentioned in the introduction, there is a model intimately related to the KPZ equation; namely, the directed polymer in a random medium. The model representation consists of a linear diffusion equation with multiplicative noise, as shown in equation (3). The spatial discretization of this equation is completely under control, as the discretization only affects the Laplacian term. The SC physics enters via the local noise term, and will not be affected by spatial discretization; the course is clear. We implement the spatial discretization via the diffusion-like equation and then make the inverse Hopf–Cole transformation on the lattice in order to obtain a discrete form of the KPZ equation which is guaranteed to be stable for arbitrary values of the coupling.

So explicitly we rewrite equation (3) as

$$dw_i/dt = (v/a^2) \sum_j (w_j - w_i) + (\lambda/v) w_i \eta_i. \tag{17}$$

We now make the inverse transformation

$$w_i(t) = \exp[\lambda h_i(t)/v] \tag{18}$$

which yields the new discrete form of the KPZ equation:

$$dh_i/dt = (v^2/a^2 \lambda) \sum_j \{ \exp[(\lambda/v)(h_j - h_i)] - 1 \} + \eta_i. \tag{19}$$

This equation appears unwieldy, but contains a very important message. It is useful to define local ‘velocities’ $v_{i,j} = F(h_i - h_j)$ in such a way that one may write the discrete KPZ equation in the form

$$dh_i/dt = \sum_j v_{i,j} + \eta_i. \tag{20}$$

One then sees that the functional form of these velocities for the above discretization scheme takes the form

$$F(x) = (v^2/\lambda a^2)[\exp(-\lambda x/v) - 1]. \quad (21)$$

In contrast, the conventional scheme as shown in equation (4) gives a functional relationship of the form

$$F(x) = -(v/a^2)x + (\lambda/2a^2)x^2. \quad (22)$$

Comparing these two forms indicates that the conventional scheme is a truncated form of the new scheme to second order in $(\lambda x/v)$ which will be accurate so long as $x \sim (h_i - h_j) < v/\lambda$ which is consistent with our findings in the previous section: the conventional scheme becomes unstable when neighbouring height differences occur which are of order v/λ . The physics of the KPZ equation in the SC regime is such that large local maxima (i.e. those points which have large positive local height differences) are *not* evolved by the equation. They experience a weak negative force of order $(v^2/\lambda a^2)$. We see that the conventional schemes fail because they actually boost such points in the positive direction. This unphysical behaviour will always be present if one attempts to discretize the KPZ equation using a finite number of terms in the height differences. The correct discretization schemes (of which equation (19) is one example) must be such as to *freeze* points which are local maxima. This effective freezing was already seen in the deterministic top hat problem studied in section 2.

To end this section we make some practical remarks. The scheme presented in equation (19) is unwieldy and will become more so when one discretizes time as well. In this case one must not discretize time in the h representation, but must inverse Hopf–Cole transform the spacetime discretized equation for w . It is clear that one may as well numerically integrate the equation for w and then perform the statistical analysis on logarithms of that function. In fact one is essentially simulating the directed polymer problem. This implies that there is no *easy* way of directly integrating the KPZ equation. The only physically meaningful way of doing this is to use equation (19), which is tantamount to dealing with directed polymers. In fact, there is one example in the literature of a direct integration of the discretized equation for w in $d = 2$ [15]. The authors were motivated to integrate this form of the equation due to the reported problems in earlier work of numerical instability for larger values of the coupling constant. This motivation has been seen in the previous sections of the current work to be well justified. Their results for the dynamic exponents agree well with those found in [10], although they stress the difficulty of determining whether their results were obtained in the true asymptotic regime. It is certainly worthwhile to extend this type of numerical analysis to both larger values of the coupling constant, and to higher dimensions.

4. New continuum analogues

Aside from the practical concerns of how to integrate the KPZ equation, the previous sections have shed some light on what the SC physics of the KPZ equation actually means. What we have seen is that for large values of the coupling constant, local maxima in the interface become frozen or pinned. The interesting dynamical effects occur when we pump such a system with noise. Only the largest positive deviations will remain pinned for a long time, the smaller ones being swamped by the local noise fluctuations. In other words, the rare fluctuations become frozen in and therefore assume a much greater statistical weight.

The question is how sturdy is this type of SC physics. Referring again to equation (21), we see that the SC physics of the KPZ equation lies in the fact that $F(x)$ is essentially zero for $x > 0$. The precise form of this function for $x < 0$ may not be so important—this part of the function simply controls the rate at which lower points try to catch up with their frozen higher neighbours. To test the hypothesis that the KPZ universality class is determined only through the special features of $F(x)$ for $x > 0$, it is interesting to consider other models which retain this structure.

One such example is the continuum model defined via

$$\partial_t h = D \nabla^2 h + \eta \quad (23)$$

where

$$D = \begin{cases} 0 & \nabla^2 h < 0 \\ D_0 & \nabla^2 h > 0. \end{cases} \quad (24)$$

This model is a not-too-distant relative of a stochastically pumped version of Barenblatt's equation [17, 18].

The deterministic evolution of this equation consists of regions of negative curvature being strongly pinned, whilst regions of positive curvature are diffusively relaxed. In this sense it is an inherently SC realization of the KPZ equation. The relaxational time scales are certainly slower in this model than in the KPZ equation, but as we argued above, if one believes that the SC evolution is dominated by frozen surface maxima, then this difference of time scales may not effect the universal features of the model. The pinning of regions of negative curvature in the above model actually makes it a prime continuum candidate for the description of the RSOS model, in which evolution is only possible within regions of positive curvature (although one must probably be more careful with the definition of the noise in this case). It would be of interest to numerically study this model and ascertain whether it shares the same scaling features as the KPZ equation. Perhaps it is even possible to perform some analytic treatment. The nice feature of this model (as with many others which may be generated by altering the function $F(x)$ for $x < 0$) is that it is intrinsically SC in the KPZ sense. We leave this speculative section now and end the paper with our more concrete conclusions.

5. Discussions and conclusions

We have shown that conventional discretization schemes [8–10] of the continuum KPZ equation (such as that shown in equation (4)) have two failings which forbid them to remain faithful to the continuum equation, especially in the SC regime. First they exhibit ghost fixed points (i.e. certain configurations which are stationary solutions) which are absent in the continuum; and secondly, these ghost fixed-points are unstable, so that the models deviate exponentially fast from the physical evolution. We believe we have clearly demonstrated these defects such that the previous results obtained from numerical integration of these schemes must be viewed as unsound. (One might hope to stabilize the equation by including longer-ranged terms in the lattice derivatives. However, the physics is such that one actually needs an infinite-ranged representation to guarantee stability—the continuum limit is stable in the vicinity of discontinuities due to the *divergence* of the continuum Laplacian.)

We have shown that by discretizing the directed polymer version of the KPZ equation, one may establish a stable, physically faithful scheme for integration of the KPZ equation. The main advantage of this new discretization is not so much its practical worth, but rather

its illumination of the true nature of SC behaviour in the KPZ equation. On the basis of this new insight we have postulated a new criterion for models to be in the continuum KPZ universality class, and have suggested perhaps the simplest continuum member of this class, which is related to the Barenblatt equation with a stochastic source.

A perplexing question remains. Although we now understand how to make discrete models faithful to the continuum KPZ equation, it is disconcerting that the discrete forms such as equation (4) are not in the continuum KPZ universality class, when they seem so reasonable at first glance. It opens the possibility of a more tenuous link between microscopic (lattice) models and the continuum KPZ equation. What we have seen is that SC physics in the KPZ equation requires local maxima to be strongly pinned. This kind of physics is transparent in some microscopic models such as the RSOS model, and polynuclear growth models. However, other microscopic models which might be better represented by naive lattice versions of the KPZ equation, will certainly not be in the true KPZ universality class. The cataloguing of the universality classes of the KPZ equation, and other non-equilibrium interface models, remains as ever an absorbing challenge for the future.

Acknowledgments

TJN is grateful to Michael Swift for useful preliminary discussions, and acknowledges financial support from the Engineering and Physical Science Research Council. AJB thanks Matteo Marsili for discussions.

Appendix A.

In this appendix we extend the results of the top hat problem studied in section 2. We shall connect the existence of a critical coupling constant to the known behaviour of the noisy KPZ problem in the vicinity of the weak coupling fixed point. First, we generalize the results to arbitrary dimension, by considering a pill-box in dimension d , of radius b and height H . Restricting our attention solely to the centre of the pill-box we have the exact solution

$$h(\mathbf{0}, t) = (v/\lambda) \ln\{1 + [\exp(\lambda h/v) - 1] I(t)\} \quad (\text{A1})$$

where

$$I(t) = \frac{2}{\Gamma(d/2)} \int_0^{b/(4vt)^{1/2}} dy y^{d-1} e^{-y^2} \quad (\text{A2})$$

and $\Gamma(z)$ is the Gamma function [12]. For $vt \gg b^2$ this expression simplifies to

$$h(\mathbf{0}, t) \sim (v/\lambda) \ln \left\{ 1 + \frac{[\exp(\lambda H/v) - 1]}{\Gamma(1 + d/2)} \frac{b^d}{(4\pi vt)^{d/2}} \right\}. \quad (\text{A3})$$

We see once again that there exists a critical value for the dimensionless parameter $(\lambda H/v)$. For $(\lambda H/v) \ll 1$ the above expression may be expanded out to give the simple diffusion result for large times. Explicitly one has

$$h(\mathbf{0}, t) \sim b^d H / (4\pi vt)^{d/2}. \quad (\text{A4})$$

However, for $(\lambda H/v) \gg 1$ we may only expand the logarithm for times larger than t^* where

$$t^* = (b^2/v) \exp(2\lambda H/dv). \quad (\text{A5})$$

For $t \gg t^*$, the centre of the block once again diffuses away. However, for $t \ll t^*$, the centre is essentially frozen, and decays only logarithmically slowly. Explicitly one has

$$h(\mathbf{0}, t) \sim H - (dv/2\lambda) \ln(vt/b^2). \tag{A6}$$

The message we wish to draw from this result is that for any finite dimension d , there exists a critical value of the coupling constant of order $\lambda_c \sim (v/H)$, such that for $\lambda \gg \lambda_c$, a surface deviation which has a scale of H will become essentially frozen, whilst a surface deviation of a smaller scale will diffuse away, almost unaware of the nonlinearity. If the reader feels uncomfortable with the discontinuous properties of the top hat function, we note that similar asymptotic results may be obtained by replacing the top hat over the range $|x| < b$ by an inverted parabola of central height H .

Now consider the full noisy KPZ equation, starting from an initially flat interface. Over an initial time period the surface fluctuations will evolve according to the linear (or Edwards–Wilkinson [19]) model. Accordingly the root-mean-square fluctuations grow in time as

$$\langle (\delta h)^2 \rangle^{1/2} \sim \begin{cases} t^{(2-d)/4} & d < 2 \\ (\ln(t/t_0))^{1/2} & d = 2 \\ \text{constant} & d > 2. \end{cases} \tag{A7}$$

It is our contention that the SC physics of the KPZ equation will come into effect only when these rms fluctuations become of the same order as a critical height scale H , which is set by the condition $\lambda H/v \sim O(1)$. This idea then leads to a prediction of the crossover time, t_c from weak to SC behaviour by equating $\langle (\delta h)^2 \rangle^{1/2}$ with $H \sim v/\lambda$. One finds for $d < 2$ that $t_c \sim \lambda^{-4/(2-d)}$. For $d = 2$ one has $t_c \sim t_0 \exp[C(v^3/D\lambda^2)]$ (where C is some number of order unity). For $d > 2$ the rms fluctuations saturate to a constant value. One then has a very simple interpretation of the weak/SC phase transition. For small λ , the critical height H is large and thus exceeds the saturated rms fluctuations—hence there will be no possibility for the SC behaviour to set in. Conversely, for large λ , the critical height H is small and will be quickly swamped by the weak rms fluctuations, such that the SC behaviour is initiated.

The particular forms for the weak/SC crossover time found above for $d \leq 2$ have been found in previous studies [20] by an explicit integration of the one-loop renormalization group flow equations. The simple derivation above is useful as it brings out the physical mechanism behind the flow to SC physics—i.e. the freezing of rms fluctuations of order v/λ and the subsequent nonlinear evolution.

Appendix B.

In this appendix, we examine the eigenvalue spectrum of the matrix \mathbf{M} that appears in the stability analysis of the N -site problem in section 2. Explicitly one has the $N \times N$ matrix

$$\mathbf{M} = \begin{pmatrix} -2 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 3 & -2 \end{pmatrix}. \tag{B1}$$

Table A1.

N	4	5	6	7	8	10	13	20	25	29
l_1	0.303	0.236	0.199	0.175	0.160	0.141	0.132	0.122	0.1215	0.1214
l_2	—	—	—	0.000	0.047	0.091	0.108	0.121	0.1212	0.1213

The interest lies in whether this matrix has positive eigenvalues (which signal an instability in the original problem), and if so, whether these eigenvalues survive in the large- N limit.

We determine the eigenvalues by direct numerical computation using the *Mathematica* program. The result is that for increasing N , there exist two positive eigenvalues which become degenerate at a value of 0.1214(1) for large- N . A sample of the results for finite N is presented in table A1.

References

- [1] Kardar M, Parisi G and Zhang Y-C 1986 *Phys. Rev. Lett.* **56** 889
- [2] Krug J and Spohn H 1991 *Solids Far From Equilibrium: Growth, Morphology and Defects* ed C Godrèche (Cambridge: Cambridge University Press)
- [3] Moore M A, Blum T, Doherty J P, Marsili M, Bouchaud J-P and Claudin P 1995 *Phys. Rev. Lett.* **74** 4257
- [4] Zabolitzky J G and Stauffer D 1986 *Phys. Rev. A* **34** 1523
- [5] Kertész J and Wolf D E 1989 *Phys. Rev. Lett.* **62** 2571
- [6] Tang L-H, Forrest B M and Wolf D E 1992 *Phys. Rev. A* **45** 7162
- [7] Kim J M and Kosterlitz J M 1989 *Phys. Rev. Lett.* **62** 2289
- [8] Amar J G and Family F 1989 *Phys. Rev. A* **41** 3399
- [9] Grossmann B, Guo H and Grant M 1991 *Phys. Rev. E* **43** 1727
- [10] Moser K, Kertész J and Wolf D E 1991 *Physics A* **178** 215
Moser K and Wolf D E 1994 *J. Phys. A: Math. Gen.* **27** 4049.
- [11] Ertas D and Kardar M 1993 *Phys. Rev. E* **48** 1228
- [12] Abramowitz M and Stegun I A (ed) 1972 *Handbook of Mathematical Functions* 10th edn (New York: Dover)
- [13] Marsili M and Bray A J 1996 *Phys. Rev. Lett.* **76** 2750
- [14] Newman T J and Swift M unpublished
- [15] Beccaria M and Curci G 1994 *Phys. Rev. E* **50** 4560
- [16] Kim J M, Bray A J and Moore M. A 1991 *Phys. Rev. A* **44** 2345
- [17] Barenblatt G I 1979 *Similarity, Self-Similarity and Intermediate Asymptotics* (New York: Consultants Bureau)
- [18] Goldenfeld N 1992 *Lectures on Phase Transitions and the Renormalization Group* (Reading, MA: Addison-Wesley)
- [19] Edwards S F and Wilkinson D R 1982 *Proc. R. Soc. A* **381** 17
- [20] Nattermann T and Tang L-H 1992 *Phys. Rev. A* **45** 7156